Intrinsic Kinematics

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Abstract

We show how some geometric elements of the path of a particle moving in a plane – the osculating circle and its radius of curvature – can be used to construct the parabolic trajectory of projectiles in motion under gravity.

Key words: Osculating Circle, Radius of Curvature, Parabolic Motion

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1 Intrinsic Equations

The acceleration vector of a material point moving in a plane can be resolved in two special directions which are independent of the choice of the particular system of reference used to describe the motion. These *intrinsic* directions are the tangent to the trajectory of the material point and the perpendicular to it in the plane of the motion [1].

Fig.1a shows the situation for a particle describing an arbitrary trajectory in the plane. At the position P of the particle we have indicated the direction of the velocity vector $\vec{v}$ and the total acceleration vector $\vec{a}$.

The component of the acceleration tangent to the path, a_t, measures the rate of change of the magnitude of the velocity vector. The component of the acceleration normal to the path, a_n, measures the rate of change of the direction of the velocity vector.

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In Fig. 1a, we have also drawn a circle of radius ρ and center O which is tangent to the path at P. When this circle fits the curve just right at P it is called the *osculating circle* of the path at that point. The osculating circle is very helpful in determining the component of the acceleration normal to the path of the particle. If we imagine that when the particle is at P, instead of following its real path, it describes a uniform motion around the osculating circle itself, the component of the acceleration normal to the path becomes the centripetal acceleration in this motion, having magnitude $a_n = v^2/\rho$ in direction of the radius PO of the osculating circle [1, 2].

Now note that in Fig.1a we have represented the total acceleration vector in the direction of the chord PQ of the osculating circle, making an angle ϕ with the radius PO. Since $a_n = a\cos\phi$ we also have

$$\frac{v^2}{\rho} = a\cos\phi. \tag{1.1}$$

The magnitude of the total acceleration of the particle in (1.1) can be related to yet another geometric element of the osculating circle, namely, the length of the chord PQ between the particle and the osculating circle, in the direction of the total acceleration vector. From Fig. 1a we see that this length is $C = 2\rho\cos\phi$. Substituting this value for ϕ into (1.1) yields:

$$C = \frac{2v^2}{a}. \tag{1.2}$$

2 Projectile Motion

Relation (1.2) finds an interesting application in the study of projectile motion under gravity [3]. In Fig. 1b we have represented the parabolic trajectory described by a projectile fired with velocity $\vec{v_0}$ at an angle θ to the horizontal. Suppose that when the projectile is at P its velocity vector $\vec{v}$ makes an angle β to the horizontal. Seeing that the horizontal projection of the motion is uniform, the equality $v\cos\beta = v_0\cos\theta$ holds at P. The acceleration in the direction of the chord PQ is g due to gravity. Thus Eq. (1.2) becomes

$$C_\beta = \frac{2v_0^2\cos^2\theta}{g\cos^2\beta}. \tag{2.1}$$

Formula (2.1) allows us to construct the parabolic motion of the projectile from the intrinsic elements developed above [4]. First we note that when $\beta = 0$ the particle reaches the *vertex* V of

the parabola. The length of the chord PQ in this position is

$$C_{(\beta=0)} = 2p = \frac{2v_0^2\cos^2\theta}{g}. \tag{2.2}$$

The above relationship determines a length p which is the distance between the *focus* and the *directrix line* of the parabola. This distance is the basis for the construction of the parabola, as we will see in the next section, since the defining property of a parabola is that any point on it is equidistant from the focus and the directrix.

3 The Parabola

We begin the construction of the parabola by tracing the line PH normal to the path along the radius of the osculating circle (But note that H is *not* the center of the circle), and PG along the horizontal, as shown in Fig.1b. The *axis* of the parabola is the vertical line through H and G when $GH = p$.

To locate the *focus* we invoke the *reflective property* of the parabola, according to which any *light ray* (PQ) parallel to the axis and incident on the parabola is reflected to the focus. The ray PQ incides on the parabola at P making an angle β with respect to PH, and is refleted to F in such a way that the angle of reflection equals the angle of incidence, or $\widehat{HPF} = \beta$.

It follows from simple trigonometric relations in the triangles PGF and PGH that

$$PH = 2PF\cos\beta \qquad \text{and} \qquad p = PH\cos\beta.$$

Eliminating PH among the above equations we get the following expression for the distance between of the particle to the focus of the parabola

$$PF = \frac{p}{2\cos^2\beta}. \tag{3.1}$$

From (3.1) we see that the vertex of the parabola is a point on its axis at a distance $p/2$ from the focus. The distance (3.1) is related the length (2.1) of the chord PQ at any point of the path of the particle by $C_\beta = 4PF$.

These elements suffice to construct the parabola (See [5] for the more usual analytical description).

Finally, we mention that two important parameters pertaining to a more physical analysis of the projectile motion are the total horizontal distance (or range) and the maximum height attained by the projectile in the case it returns to the same horizontal it was launched at. We work out the expression for the range here, and leave it to the reader to figure out how to use our analysis to determine the maximum height.

The range R is just twice the horizontal distance PG when P is the launching point of the projectile, and G is a point on same the horizontal from P. In this case PG forms an angle $\pi/2 - \theta$ with the corresponding line PH, and we obtain

$$R = 2p \tan\theta = \frac{v_0^2 \sin 2\theta}{g}. \tag{3.2}$$

References

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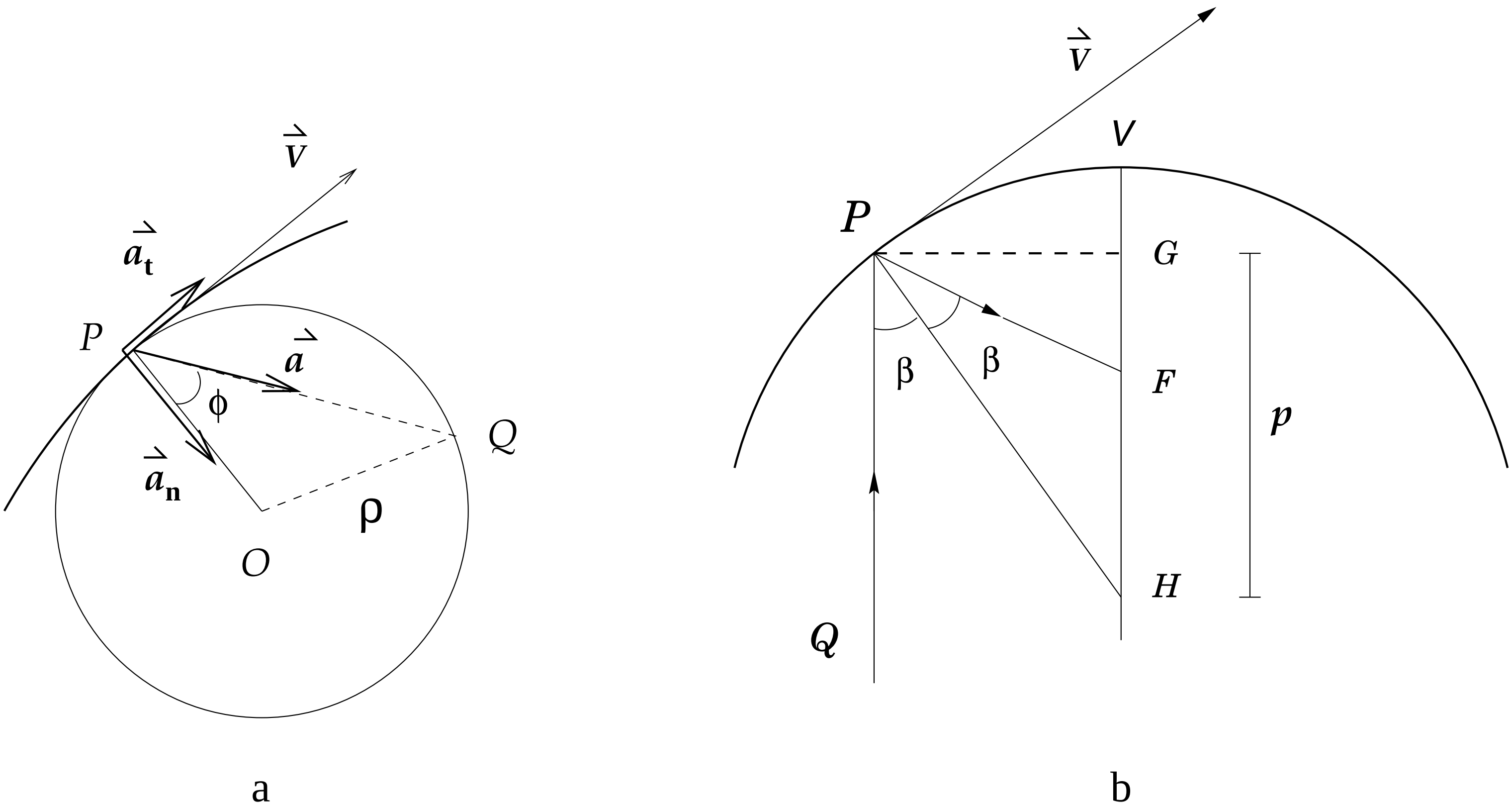

Figure 1: (a) Arbitrary plane motion, and (b) parabolic motion.